A Wall Law for Turbulent Boundary Layers in Adverse Pressure Gradients

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A relation between mean velocity and turbulent shear stress has been derived for turbulent boundary layers developed under adverse pressure gradients. The wall law has been obtained by performing a one-dimensional analysis on the turbulent kinetic energy equation with assumptions of local similarity. The analysis relates the diffusional flux to the shear stress gradient. Physical limiting conditions and known experimental results are used to evaluate unknown constants appearing in the derived equation. The resulting wall law is particularly suited for the near-wall treatment in the numerical calculations.

Nomenclature

B, B^*, c_D, κ_0	= empirical constants
J	= diffusional flux of k
k	= turbulent kinetic energy
p	= pressure
u, u^+	= mean velocity in x direction
x	= streamwise coordinate
y, y^+, η	= coordinates normal to the wall
α	= stress (pressure) gradient parameter
ϵ	= dissipation rate
κ, κ^*	= von Kármán constant and modified von Kármán constant
ν	= kinematic viscosity
ρ	= density
σ	= Prandtl number for k
τ	= shear stress

Superscripts and Subscripts

s	= slip values
w	= wall
$+$	= dimensionless quantities based on $(\tau_w/\rho)^{1/2}$ and $\nu/(\tau_w/\rho)^{1/2}$

Introduction

IN many practical situations, turbulent boundary layers develop in adverse pressure gradients. In such boundary layers, the shear stress near the wall increases in the direction normal to the wall so as to be in balance with the pressure increase in the streamwise direction. For the separated flows, the shear stress vanishes at the wall and increases to its peak, which is located relatively far from the wall.

Townsend (Ref. 1, pp. 176-184) considered the balance of turbulent kinetic energy, and obtained a velocity profile under an assumption of the linear shear stress distribution. The concept of structural similarity led Townsend to relate the diffusional flux to the 3/2 power of the local shear stress rather than to its gradient. As a result, Townsend had to introduce a free parameter associated with the diffusional flux, which inconveniently varies from zero to a certain positive value corresponding to the constant stress layer and the zero wall stress layer.

The present paper proposes a new version of the one-dimensional analysis on the turbulent kinetic energy balance equation and derives a general wall law for the velocity-stress relation appropriate to the turbulent boundary layers in adverse pressure gradients. The diffusional flux is related to the gradient of the turbulent kinetic energy following the effective viscosity formulation. The unknown constants appearing in the derived equation are evaluated using available experimental results and physical limiting conditions in consideration of the constant stress layer and the zero wall stress layer.

The velocity profiles are generated from the present velocity-stress relation and plotted for the linear stress distributions so as to compare the present theory with existing analyses and available experimental data. The discussion further extends to the turbulent kinetic energy budget.

Modeling on Turbulent Kinetic Energy Equation

For turbulence in the vicinity of a wall, the convection of the turbulent kinetic energy is nearly always negligible. Thus, in the present analysis, the energy convection terms are dropped. Moreover, all dependent variables are supposed to be functions only of the distance from the wall. Then, the conservation equation for the turbulent kinetic energy may be written in the boundary-layer coordinates (x,y) as

$$-\frac{\mathrm{d}J}{\mathrm{d}y}+\tau\frac{\mathrm{d}u}{\mathrm{d}y}-\rho\epsilon=0 \tag{1}$$

where τ is the local shear stress and J the diffusional flux of the turbulent kinetic energy k, while its rates of production and dissipation are presented by the terms $\tau\,(\mathrm{d}u/\mathrm{d}y)$ and $\rho\epsilon$, respectively.

The present analysis employs the gradient diffusion concept rather than the one proposed by Townsend for reasons that will become apparent as the analysis proceeds. (Certain experimental data, e.g., Ref. 2, do suggest the gradient diffusion mechanism.) Thus, the eddy viscosity formulation is adopted for the simulation of flux J as

$$J=-\frac{1}{\sigma}\frac{\tau}{\mathrm{d}u/\mathrm{d}y}\frac{\mathrm{d}k}{\mathrm{d}y}=-\frac{\tau}{\sigma}\frac{\mathrm{d}k}{\mathrm{d}u}=-\frac{1}{\rho\sigma c_D^{1/2}}\tau\frac{\mathrm{d}\tau}{\mathrm{d}u} \tag{2}$$

where σ is the effective Prandtl number for k. In this equation, the local similarity assumption between k and τ, namely, $\tau=\rho c_D^{1/2}k$, has been used with the near-wall constant $c_D\cong0.09$. This approximate relation, developed by Townsend (Ref. 1, pp. 120-122), is reasonably borne out by experiments on wall turbulence. Equation (2) naturally conforms to the local equilibrium condition $\tau\,(\mathrm{d}u/\mathrm{d}y)=\rho\epsilon$ in the case of the

Received April 13, 1983; revision received Nov. 18, 1983.

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constant stress layer,

$$J=0 \qquad \text{for } \tau=\tau_w=\text{const} \tag{3}$$

where the subscript w refers to the wall, such as τ_w for the wall shear stress.

In the case of the zero wall stress layer, the present gradient-type expression for J transforms itself into Townsend's proposal for the flux. This can readily be shown by considering Eq. (2) for the limiting case of a zero wall stress layer corresponding to the incipient separation. Stratford[3] found that the velocity in the zero wall stress layer varies in proportion to the square root of the distance from the wall. Since, in such a stress layer, the shear stress vanishes at the wall and increases linearly away from the wall, Eq. (2) reduces to

$$J=-\frac{\rho\kappa_0}{\sigma c_D^{1/2}}\left(\frac{\tau}{\rho}\right)^{3/2} \quad \text{for } \tau=\rho\left(\frac{\kappa_0 u}{2}\right)^2 \tag{4}$$

where κ_0 is an empirical constant. Observations by Stratford[3] suggest $\kappa_0 \cong 0.5$.

Equations (3) and (4) together indicate the following relations corresponding to the two limiting cases:

$$\rho\epsilon=\tau\frac{du}{dy} \qquad \text{for } \frac{\tau}{\tau_w}=1 \tag{5a}$$

$$\rho\epsilon=\left(1+\frac{3}{2}\frac{\kappa_0^2}{\sigma c_D^{1/2}}\right)\tau\frac{du}{dy} \qquad \text{for } \frac{\tau}{\tau_w}\to\infty \tag{5b}$$

The above observation on the limiting cases may prompt one to simulate the dissipation rate ϵ as

$$\rho\epsilon-\tau\frac{du}{dy}=\frac{\kappa^{*2}}{\sigma c_D^{1/2}}\left(\frac{\tau-\tau_w}{\tau}\right)^n\tau\frac{du}{dy} \tag{6a}$$

and

$$\kappa^*|_{\tau_w\to 0}=(3/2)^{1/2}\kappa_0 \tag{6b}$$

where

$$n>0 \tag{6c}$$

The parameter κ^* has been introduced in Eq. (6a). The only information available at this moment is its asymptotic behavior, as indicated by Eq. (6b).

Combining Eqs. (6a) and (2) transforms the turbulent kinetic energy equation (1) into the following differential equation:

$$\frac{d}{dy}\tau\frac{d\tau}{du}=\rho\kappa^{*2}\left(\frac{\tau-\tau_w}{\tau}\right)^n\tau\frac{du}{dy} \tag{7}$$

In order to investigate the unknown exponent n, one may linearize Eq. (7) by perturbing τ as

$$\tau=\tau_w+\tau_w' y \tag{8a}$$

where

$$\tau_w'\equiv\left(\frac{d\tau}{dy}\right)_w \tag{8b}$$

and integrate the resulting linearized equation once to obtain

$$\frac{du}{dy}\simeq\frac{1}{\kappa^* y}\left(\frac{1+n}{2}\frac{\tau_w}{\rho}\right)^{1/2}\left(\frac{\tau_w}{\tau_w' y}\right)^{(n-1)/2} \tag{9}$$

Since the local velocity gradient du/dy must remain finite even for the limiting case of the constant stress layer, i.e., $\tau_w' y/\tau_w\to 0$, the exponent n must be unity. Then, Eq. (9) automatically reduces to the conventional "law of the wall,"

$$\frac{du}{dy}=\frac{1}{\kappa^* y}\left(\frac{\tau_w}{\rho}\right)^{1/2} \tag{10a}$$

and

$$\kappa^*|_{\tau_w'\to 0}=\kappa \text{ (von Kármán const)} \tag{10b}$$

Because of the asymptotic behavior indicated by Eq. (10b), the parameter κ^* will be called the "modified von Kármán constant" hereafter.

Velocity-Stress Relation

Upon substituting $n=1$ into Eq. (7), one obtains the following general differential expression, which is valid for a turbulent boundary layer with the shear stress increasing from the wall:

$$\frac{d}{d\tau}\left(\tau\frac{d\tau}{du}\right)^2=2\rho\kappa^{*2}(\tau-\tau_w)\tau \tag{11}$$

It is interesting to note that the independent variable y has been eliminated from the equation. The right-hand side of Eq. (11) may now be integrated in terms of τ with the boundary condition $[\tau\,(d\tau/du)]_w^2=0$. The result is given in the dimensionless form as

$$\frac{du^+}{d\tau^+}=\frac{1}{\kappa^*}\frac{\tau^+}{(\tau^+-1)[(1+2\tau^+)/3]^{1/2}} \tag{12a}$$

where

$$u^+=u/(\tau_w/\rho)^{1/2} \tag{12b}$$

and

$$\tau^+=\tau/\tau_w \tag{12c}$$

The integration of Eq. (12a) yields the following velocity-stress relation, which is of primary interest in the present analysis:

$$u^+=\frac{1}{\kappa^*}\left[3(t-t_s)+\ln\left(\frac{t_s+1}{t_s-1}\frac{t-1}{t+1}\right)\right] \tag{13a}$$

where

$$t=\left(\frac{1+2\tau^+}{3}\right)^{1/2} \tag{13b}$$

The slip value t_s has been introduced in Eq. (13a). In turbulent boundary layers with adverse pressure gradients, the shear stress near the wall is found to vary linearly as

$$\tau^+=1+\alpha y^+ \tag{14a}$$

where

$$\alpha\equiv\nu\rho^{1/2}\tau_w'/\tau_w^{3/2} \tag{14b}$$

$$y^+=(\tau_w/\rho)^{1/2}y/\nu \tag{14c}$$

and α is the parameter representing the shear stress gradient. The slip value t_s may be determined from the condition that u^+ vanishes deep within the sublayer at $y^+=y_s^+$. Upon substituting Eq. (14a) into Eq. (13a) and assuming $\alpha y^+\ll 1$, it can readily be shown that one of the limiting cases indeed leads to the "logarithmic law," namely,

$$u^+|_{\alpha\to 0}=(1/\kappa)\ln(y^+/y_s^+) \tag{15}$$

The comparison of this equation with the conventional "log law" implies that

$$y_s^+\simeq e^{-\kappa\beta} \tag{16}$$

The substitution of Eq. (16) into Eq. (13b) gives the following expressions for the slip value t_s:

$$t_s(\alpha) = (1 + \tfrac{2}{3} e^{-\kappa B}\alpha)^{1/2} \simeq (1+0.074\alpha)^{1/2} \tag{17}$$

where the wall law intercept B and the von Kármán constant κ are assumed to be 5.5 and 0.4, respectively. Similarly, it can be shown that the other limiting case implicit in Eq. (13a) corresponds to the zero wall stress layer and

$$u^+ |_{\alpha\to\infty} = \frac{2}{\kappa_0}\alpha^{1/2}(y^{+1/2} - y_s^{+1/2}) \simeq \frac{2}{\kappa_0}(\alpha y^+)^{1/2} \tag{18}$$

From the information on the asymptotic behavior represented by Eqs. (6b) and (10b), a rough estimation on the modified von Kármán constant κ^* may be made by

$$\kappa^*(\alpha) = \frac{\kappa + (3/2)^{1/2}\kappa_0\alpha}{1+\alpha} \simeq \frac{0.4+0.6\alpha}{1+\alpha} \tag{19}$$

As indicated above, the modified von Kármán constant, which is the only parameter yet to be specified empirically, varies in a narrow range (0.4 ~ 0.6), while one must vary the diffusion constant from zero to a certain positive value corresponding to the constant stress layer and zero wall stress layer if one follows Townsend's proposal for the diffusional flux. This fact implies that the present gradient-type model describes the actual diffusion process better than the one proposed by Townsend.

The analysis developed thus far is quite general in the sense that the velocity is related directly with the local shear stress instead of with the normal distance from the wall. Although the linear stress distribution is assumed in consideration of the limiting cases, the present velocity-stress relation, namely, Eqs. (13) with Eqs. (17) and (19), may be applied for any monotonously increasing shear stress layers and need not be restricted to the linear stress layer discussed below.

Discussion

Since the convection terms vanish near to the wall, the equation of motion may be written as

$$\frac{d\tau}{dy} = \frac{dp}{dx} \tag{20}$$

which can be integrated to yield the linear stress distribution,

$$\tau^+ = 1 + \eta \tag{21a}$$

where

$$\eta = \alpha y^+ \tag{21b}$$

and

$$\alpha = \nu\rho^{1/2}\frac{dp}{dx}\Big/\tau_w^{3/2} \tag{21c}$$

Under this linear stress approximation, the production and diffusion rates are plotted in the Fig. 1 vs the parameter η for all values of α. In the figure, the coefficient associated with the diffusion rate ($\kappa^{*2}/\sigma c_D^{1/2}$) is denoted simply by Dp. Moreover, the superscript + refers, as before, to the dimensionless quantities based on the reference velocity $(\tau_w/\rho)^{1/2}$ and the reference length $\nu/(\tau_w/\rho)^{1/2}$.

The turbulent length scale defined by $\ell \equiv (\tau/\rho)^{1/2}/(du/dy)$ is also indicated in Fig. 1. The dimensionless turbulent length scale increases almost linearly, but asymptotically changes its slope by a factor of $(2/3)^{1/2}$. The gain rate of the diffusion increases monotonously, while that of the production has its minimum value at $\eta = 3^{1/2}$.

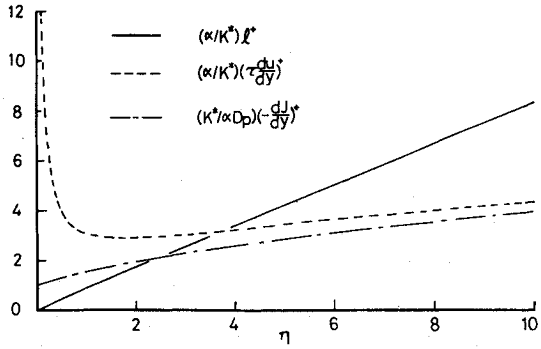

Fig. 1 Turbulent length scale and kinetic energy budget.

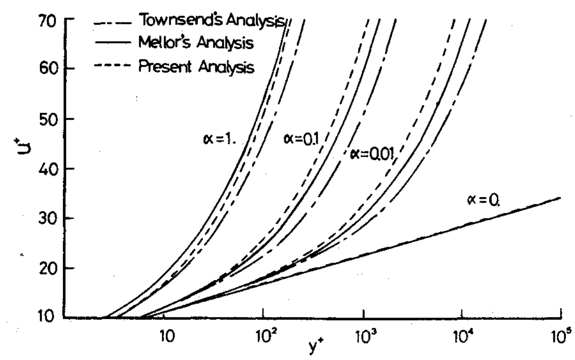

Fig. 2 Mean velocity profiles for various pressure gradients.

Townsend (Ref. 1, pp. 176-184) assumed the linear stress distribution and derived the wall law given by

$$u^+ = \frac{1}{\kappa}\left[2(1-B^*)(S-S_s) + \ln\left(\frac{S_s+1}{S_s-1}\,\frac{S-1}{S+1}\right)\right] \tag{22a}$$

where

$$S = (1+\eta)^{1/2} = (1+\alpha y^+)^{1/2} \tag{22b}$$

and

$$S_s = (1+0.111\alpha)^{1/2} \tag{22c}$$

As indicated earlier, the parameter B^*, introduced for the diffusional flux on the basis of the structural similarity, has not been correlated with the pressure gradient parameter α. Townsend, however, suggested $B^* \cong 0.2$ as a result of experimental data obtained by Schubauer and Klebanoff.[4]

Through an entirely different approach, Mellor[5] derived a wall law that happens to correspond to Townsend's with $B^* = 0$ (Mellor's original formulation does not require slip values, but takes into account the viscous sublayer by introducing a function that vanishes outside the sublayer.) The similarity between Mellor's wall law and the present equations (13) is interesting. However, it should be noted that they stem from origins that are significantly different. Mellor's wall law is based essentially on the mixing length hypothesis with a linear length scale variation, while the present velocity-stress relation has been derived from the turbulent kinetic energy balance equation with the diffusional flux and dissipation rate modeled with consideration of the two limiting cases and is not restricted to the linear stress layers.

Figure 2 compares the present wall law with those of Townsend ($B^* = 0.2$) and Mellor ($B^* = 0$) under the linear stress approximation. The velocity levels turn out to be in fair agreement. All three functions approach to the "log law" as α diminishes to zero. An extensive comparison of the present

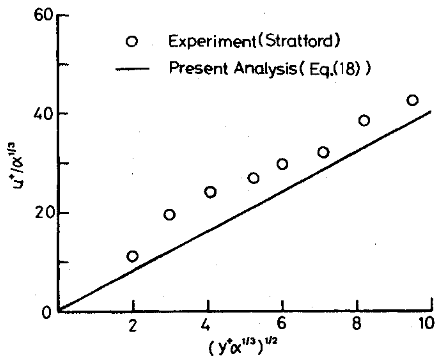

Fig. 3 Comparison of theory and experiment.

theory with experimental data will not be made here, since, for most of the existing experimental data, the values of the pressure gradient parameter α are not readily available. However, Stratford[6] provided a remarkable set of expermental data for one of the limiting cases. In Fig. 3, the data obtained by Stratford for the zero wall stress layer are compared with the present formula, namely, Eq. (18). Since the wall shear stress vanishes in this limiting case, the ordinate and abscissa variables are chosen to be $u^+/\alpha^{1/3}$ and $(y^+\alpha^{1/3})^{1/2}$, respectively. There seems to be good agreement between experiment and theory.

Conclusions

As already emphasized, the wall law derived here is quite general. However, the present theory may fail to be valid for the outer layer far from the wall, since this analysis, like most of other one-dimensional analyses, neglects the convection effects.

In the numerical calculation procedures, it has been the practice to employ the law of the wall at the near-wall nodes so as to preclude the need for fine grids. Since, in the case of an adverse pressure gradient, the shear stress increases outward normal to the wall, the conventional law of the wall based on the constant stress layer assumption cannot be employed and should be replaced with a better formula for a more accurate prediction of the boundary-layer separation. The velocity-stress relation developed in this study is particularly suited for the general wall law in the near-wall treatment, since such turbulent flow calculation schemes (e.g., Refs. 7 and 8) usually simulate the diffusion fluxes following the eddy viscosity concept and naturally become compatible with the present one-dimensional analysis near the wall.

Acknowledgments

The authors would like to express their sincere thanks to one of the reviewers of the original manuscript for many valuable constructive suggestions concerning the comparison of theories and experiments.

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